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On the Acoustic Radiation Pressure on Spheres

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Section 1—Introduction

Although frequent reference is made to acoustic radiation pressure in treatises and memoirs on sound, there appears to be no systematic theoretical development of the subject enabling actual pressures on obstacles of simple geometrical form to be calculated. In the audible range of acoustic frequencies, it is possible to devise, in a number of ways, means of measuring pressure amplitudes in sound waves as first-order effects. At supersonic frequencies, however, these methods are no longer serviceable. When the dimensions of resonators of diaphragms become comparable with the wave-length, the physical effects which enable the pressure amplitude to be measured involve intractable diffraction problems, while the extremely high frequencies and small amplitudes involved make the employment of stroboscopic methods of observation extremely difficult.

It has been shown, however, that at supersonic frequencies the acoustic radiation pressures on spheres and discs become sufficiently large to be measured easily, at any rate, in liquids. The mean pressure is generally assumed to be proportional to the energy density in the neighbourhood of the obstacle, and on this basis relative measurements can be made, for instance, in the radiation field of a supersonic oscillator.* Such formulæ may be obtained without restriction as to wave-length, for spheres in plane progressive and stationary radiation fields, and the magnitude of the pressure is found to be of entirely different orders of magnitude in the two cases.

In stationary radiation fields, the magnitude of the radiation pressure is found to be sufficiently large to account, in part, for the formation of the well-known dust-figures observed in resonance tubes filled with gas and, in particular, to explain the main features of dust striations in supersonic radiation fields in water observed by Boyle and his co-workers.

In the present paper the effect of the compressibility of the spheres and the viscosity of the medium are not taken into account, although

* Boyle and Lehmann, 'Can. J. Res.,' vol. 3, p. 491 (1930).

the analysis may be extended to include these factors. The formulæ for rigid spheres in a frictionless medium may, however, be expected to give correct orders of magnitude, and, in particular, to enable spherical torsion balances to be designed for optimum sensitivity in the measurement of radiation pressures. Should such instruments prove to be suitable for sound measurements, the procedure outlined in the present paper may be extended to obtain more refined formulæ including the compressibility of the spheres and viscosity of the medium.

Section 2—Pressure in Compressible Fluid

If we denote by ρ the density of the medium, p the pressure intensity, (u, v, w) the velocity components, the equations of motion are

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}, \qquad \rho\frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}, \qquad \rho\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z}, \tag{1}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

It is convenient to introduce ϖ defined by

$$\varpi = \int\frac{dp}{\rho}, \tag{2}$$

in terms of which the equations of motion may be rewritten,

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial\varpi}{\partial x}, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial\varpi}{\partial y}, \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial\varpi}{\partial z}. \tag{3}$$

When the motion is irrotational, we have, in terms of the velocity potential ϕ,

$$u = -\frac{\partial\phi}{\partial x}, \qquad v = -\frac{\partial\phi}{\partial y}, \qquad w = -\frac{\partial\phi}{\partial z}. \tag{4}$$

The equations of motion (3) are, then, completely satisfied if

$$\varpi = \int\frac{dp}{\rho} = \dot{\phi} - \tfrac{1}{2}(u^2 + v^2 + w^2) = \dot{\phi} - \tfrac{1}{2}q^2, \tag{5}$$

where, as usual, we denote $q^2 = u^2 + v^2 + w^2$.

We have, in addition, the equation of continuity,

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0, \tag{6}$$

which may be written

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = \nabla^2\phi. \tag{7}$$

For a medium in which $dp/d\rho = c^2$, a constant, the exact differential equation for ϕ is easily found to be

$$\frac{1}{c^2}\frac{D^2\phi}{\mathrm{D}t^2} + \frac{1}{2c^2}\frac{\mathrm{D}}{\mathrm{D}t}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\} = \nabla^2\phi. \tag{8}$$

In acoustic problems, we may usually neglect the ratio q^2/c^2 in which circumstances ϕ may be obtained as an appropriate solution of the wave-equation

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}. \tag{9}$$

More generally, we consider a medium in which p is a function of ρ only (barotropic fluid), and write

$$p = f(\rho). \tag{10}$$

In terms of the condensation $s = (\rho - \rho_0)/\rho_0$, we have the expansion

$$p = f(\rho_0 + s\rho_0) = f(\rho_0) + s\rho_0 f'(\rho_0) + \tfrac{1}{2}s^2\rho_0^2 f''(\rho_0) + \dots, \tag{11}$$

so that

$$dp = \rho_0(f' + s\rho_0 f'' + \dots)\,ds,$$

while

$$\rho^{-1} = \rho_0^{-1}(1 - s + s^2 - \dots).$$

Thus,

$$\varpi = \dot{\phi} - \tfrac{1}{2}q^2 = \int\frac{dp}{\rho} = sf' + \tfrac{1}{2}s^2(\rho_0 f'' - f') + \dots, \tag{12}$$

no constant of integration being required since s and ϖ vanish together.

Solving (12) for s in terms of ϖ we find,

$$s \sim \frac{\varpi}{f'} - \tfrac{1}{2}\frac{(\rho_0 f'' - f')}{f'}\frac{\varpi^2}{f'^2} + \dots,$$

and substituting in (11),

$$p - p_0 = \rho_0 f'\left\{\frac{\varpi}{f'} - \tfrac{1}{2}\frac{(\rho_0 f'' - f')}{f'}\frac{\varpi^2}{f'^2}\right\} + \tfrac{1}{2}\rho_0^2\frac{f''}{f'^2}\varpi^2 + \dots,$$

or,

$$p - p_0 = \rho_0\varpi + \tfrac{1}{2}\frac{\rho_0}{c^2}\varpi^2 + \dots, \tag{13}$$

where we have written $c^2 = f'(\rho_0)$.

Finally, in terms of $\dot{\phi}$ and q^2, equation (13) gives the pressure variation in the medium

$$\delta p = p - p_0 = \rho_0 \dot{\phi} + \tfrac{1}{2} \frac{\rho_0}{c^2} \dot{\phi}^2 - \tfrac{1}{2}\rho_0 q^2. \tag{14}$$

The equation (14) is correct to terms of the order q^2/c^2 and in these circumstances it is sufficiently accurate to calculate ϕ from the approximate wave equation (9). When, however, we proceed to obtain the pressure variation over the surface of a rigid obstacle, the boundary conditions require us to refer ϕ to an origin suitably placed with reference to the boundary. The first order term in (14) integrated over the boundary will, unless the obstacle is fixed, lead to dynamical equations of motion, as a result of which the obstacle performs small linear and angular oscillations. If $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ are the velocities of translation thus determined, ϕ is referred to a moving origin and

$$\dot{\phi} = \frac{D\phi}{Dt} - \dot{\xi}\frac{\partial\phi}{\partial x} - \dot{\eta}\frac{\partial\phi}{\partial y} - \dot{\zeta}\frac{\partial\phi}{\partial z} = \frac{D\phi}{Dt} + u\dot{\xi} + v\dot{\eta} + w\dot{\zeta}. \tag{15}$$

The last three terms give rise to second-order contributions to the pressure variation over the boundary and are of the same order of magnitude as the last two terms of (14).*

In the following sections we use (14) and (15) to calculate the mean resultant pressure on a rigid sphere free to move under the influence of a prescribed radiation field. The final result is correct to the order q^2/c^2. Higher accuracy for acoustic radiation pressure in waves of

* If we identify $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ with the motion (u, v, w) of a particle of the medium, $\dot{\phi} = D\phi/Dt + q^2$, so that (14) becomes $\delta p = \rho_0 D\phi/Dt + \frac{1}{2}(\rho_0/c^2)\dot{\phi}^2 + \frac{1}{2}\rho_0 q^2$, giving the pressure variation in the medium at a point which partakes of the motion of the medium. Taking time averages, the first term drops out while the last two give the average total density of energy in the medium. This is the theorem due to Langevin, quoted by P. Biquard ('Rev. d'Acoust.,' vol. 1, p. 93 (1932)). In a frictionless fluid, the velocity of a point on the boundary is not equal to the velocity of a particle immediately in contact with it, so that Langevin's theorem as it stands is not applicable to the calculation of the mean pressure components on an obstacle. The equation (14) is, however, employed by Kotani ('Proc. Phys. Math. Soc. Japan,' vol. 15, p. 32 (1933)), who calculated therefrom the mean pressure on a *rigidly fixed* circular disc. When the obstacle is free to move under the influence of the incident sound waves, the pressure contribution arising from a moving origin according to (15) cannot be omitted.

[*Note added in proof, October 13th,* 1934.—In a later paper ('Rev. d'Acoust.,' vol. 1, p. 315), Biquard continuing his edition of Langevin's lectures ('Collège de France,' 1923), arrives at the fundamental formula (14) suitable for the calculation of acoustic radiation pressures on rigid or freely suspended rigid obstacles.]

finite amplitude would require a determination of ϕ from an exact equation of the type (5) adapted to the condition $p = f(\rho)$, using in addition an expression for the pressure variation δp of the form (14) carried to higher approximations.

Section 3—Boundary Conditions

We confine ourselves for the present to dealing with a rigid spherical obstacle of mass M and mean density ρ_1, so that $M = \frac{4}{3}\pi a^3 \rho_1$. If the incident radiation field possesses radial symmetry with respect to the sphere, it will perform small oscillations along the z-axis, the dynamical equation of motion being

$$2\pi a^2 \int_0^{\pi} \delta p \cos\theta \sin\theta \, d\theta = -M\ddot{\zeta}, \tag{16}$$

where δp is given by

$$\delta p = \rho_0 \dot{\phi} - \tfrac{1}{2}\rho_0 q^2 + \tfrac{1}{2}\frac{\rho_0}{c^2}\dot{\phi}^2. \tag{17}$$

If $D\phi/Dt$ refers to an origin at the centre of the sphere moving with velocity $\dot{\zeta}$, we have, according to (15),

$$\dot{\phi} = \frac{D\phi}{Dt} - \dot{\zeta}\cos\theta\frac{\partial\phi}{\partial r} + \dot{\zeta}\frac{\sin\theta}{r}\frac{\partial\phi}{\partial\theta}. \tag{18*}$$

To determine the motion of the sphere, it is sufficiently accurate to take into account first order terms in δp, so that on integration with respect to t, we have the dynamical equation

$$2\pi a^2 \rho_0 \int_{-1}^{1} \phi\mu \, d\mu = -M\dot{\zeta}, \tag{19}$$

where, as usual, we denote $\mu = \cos\theta$.

Since the fluid must remain in contact with the sphere, we also have,

$$-(\partial\phi/\partial r)_{r=a} = \dot{\zeta}\cos\theta. \tag{20}$$

* Lamb, 'Hydrodynamics,' §. 92, p. 124 (1932).

Section 4—Solution of the Wave Equation

It is well known that the solution of the wave equation, referred to the centre of the sphere as origin may be expressed in the form

$$\phi = \sum_{n=0}^{\infty} \{A_n \psi_n(\kappa r) + A'_n f_n(\kappa r)\} (\kappa r)^n S_n, \qquad (21)^*$$

where, as usual, the frequency f of the wave is given by $\omega = 2\pi f$ and $\kappa = \omega/c$. S_n is a surface-harmonic of order n, while $\psi_n(\kappa r)$ and $f_n(\kappa r)$ are spherical wave functions, the latter being appropriate to the expression for a divergent wave. The following well-known properties of these functions are used in the sequel :—

$$\psi_n(\zeta) = \zeta^{-n}\left(\frac{\pi}{2\zeta}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(\zeta), \quad \phi_n(\zeta) = (-1)^n \zeta^{-n}\left(\frac{\pi}{2\zeta}\right)^{\frac{1}{2}} J_{-n-\frac{1}{2}}(\zeta) \qquad (22)$$

$$\psi_n(\zeta) = \frac{1}{1.3\ldots.(2n+1)}\left\{1 - \frac{\zeta^2}{2(2n+3)} + \frac{\zeta^4}{2.4(2n+3)(2n+5)} - \ldots\right\} \qquad (23)$$

$$\phi_n(\zeta) = \frac{1.3\ldots.(2n-1)}{\zeta^{2n+1}}\left\{1 - \frac{\zeta^2}{2(1-2n)} + \frac{\zeta^4}{2.4.(1-2n)(3-2n)} - \ldots\right\} \qquad (24)$$

The spherical wave function appropriate to a diverging wave is,

$$f_n(\zeta) = \phi_n(\zeta) - i\psi_n(\zeta) = \left(-\frac{1}{\zeta}\frac{d}{d\zeta}\right)^n \frac{e^{-i\zeta}}{\zeta}, \qquad (25)$$

* The principal properties of these functions are given in Lamb's 'Hydrodynamics,' 6th ed., 292, p. 503 (1932). See also Bateman, 'Partial Differential Equations,' 651, p. 384 (1932). Three types of notation are employed, as shown in the following table :—

Lamb	Bateman	Bessel-function notation
$\psi_n(x)$	$\frac{1}{x^{n+1}}\psi_n(x)$	$\frac{1}{x^{n+1}}(\frac{1}{2}\pi x)^{\frac{1}{2}} J_{n+\frac{1}{2}}(x)$
$\Psi_n(x) = \phi_n(x)$	$\frac{1}{x^{n+1}}\chi_n(x)$	$\frac{(-1)^n}{x^{n+1}}(\frac{1}{2}\pi x)^{\frac{1}{2}} J_{-n-\frac{1}{2}}(x) = -\frac{1}{x^{n+1}}(\frac{1}{2}\pi x)^{\frac{1}{2}} Y_{n+\frac{1}{2}}(x)$
$f_n(x)$	$-\frac{i}{x^{n+1}}\zeta_n(x)$	$-\frac{i}{x^{n+1}}(\frac{1}{2}\pi x)^{\frac{1}{2}} H^{(2)}_{n+\frac{1}{2}}(x)$

In this paper Lamb's notation is followed with the exception that $\phi_n(x)$ is written for $\Psi_n(x)$ as being more easily distinguished from $\psi_n(x)$ in writing and in print.

the expansion being

$$f_n(\zeta) = \frac{i^n e^{-i\zeta}}{\zeta^{n+1}} \left\{ 1 + \frac{n(n+1)}{2(i\zeta)} + \frac{(n-1)(n).(n+1)(n+2)}{2.4(i\zeta)^2} + \dots \right. \\ \left. + \frac{1.2.3.\dots(2n-1)}{2.4.6.\dots \quad 2n} \frac{1}{(i\zeta)^n} \right\}. \quad (26)$$

Considerable use is made of the following recurrence formulæ satisfied by all the functions $\psi_n(\zeta)$, $\phi_n(\zeta)$, $f_n(\zeta)$:

$$\left.\begin{aligned} \psi'_n(\zeta) &= -\zeta\psi_{n+1}(\zeta) \\ \phi_n(\zeta)\psi_{n-1}(\zeta) - \psi_n(\zeta)\phi_{n-1}(\zeta) &= 1/\zeta^{2n+1} \\ \phi_{n+1}(\zeta)\psi_{n-1}(\zeta) - \psi_{n+1}(\zeta)\phi_{n-1}(\zeta) &= (2n+1)/\zeta^{2n+3} \end{aligned}\right\}. \quad (27)$$

From the foregoing we easily see that the appropriate form for the velocity potential ϕ is, writing $\alpha = \kappa a$,

$$\phi = \sum_{n=0}^{\infty} A_n \frac{\{F_n(\alpha)\psi_n(\kappa r) - G_n(\alpha)\phi_n(\kappa r)\}}{F_n(\alpha) - iG_n(\alpha)} (\kappa r)^n P_n(\mu), \quad (28)$$

while the incident velocity potential is

$$\phi_i = \sum_{n=0}^{\infty} A_n \psi_n(\kappa r) . (\kappa r)^n P_n(\mu). \quad (29)$$

In order to satisfy the boundary conditions (20), we easily find, on making use of (27),

$$F_n(\alpha) = \alpha^2\phi_{n+1} - n\phi_n, \quad G_n(\alpha) = \alpha^2\psi_{n+1} - n\psi_n, \quad (n \neq 1), \quad (30)$$

where, for brevity we have omitted the argument α on the right-hand side of the above equations.

For $n = 1$, the dynamical equation (19) requires that

$$F_1(\alpha) = \alpha^2\phi_2 - (1 - \rho_0/\rho_1)\phi_1, \quad G_1(\alpha) = \alpha^2\psi_2 - (1 - \rho_0/\rho_1)\psi_1. \quad (31)$$

On making use of the recurrence equations (27), we easily find that for all values of n, including $n = 1$

$$F_n(\alpha)\psi_n(\alpha) - G_n(\alpha)\phi_n(\alpha) = 1/\alpha^{2n+1}, \quad (32)$$

while

$$F_{n+1}(\alpha)G_n(\alpha) - F_n(\alpha)G_{n+1}(\alpha) = \{\alpha^2 - n(n+2)\}/\alpha^{2n+3}, \quad (n \neq 1). \quad (33)$$

In particular

$$\left.\begin{aligned} F_1(\alpha)G_0(\alpha) - F_0(\alpha)G_1(\alpha) &= 1/\alpha \\ F_2(\alpha)G_1(\alpha) - F_1(\alpha)G_2(\alpha) &= \{\alpha^2 - 3(1 - \rho_0/\rho_1)\}/\alpha^5 \end{aligned}\right\}. \quad (34)$$

We now have, on making use of (19), the velocity of the centre of the sphere,

$$\dot{\zeta} = -\frac{A_1}{\alpha^3}\kappa\frac{\rho_0}{\rho_1}\frac{1}{F_1 - iG_1}. \tag{35}$$

At the surface of the sphere, the expression for the velocity potential takes the particularly simple form

$$\phi = \sum_{n=0}^{\infty}\frac{A_n}{\alpha^{n+1}}\frac{P_n(\mu)}{F_n(\alpha) - iG_n(\alpha)}. \tag{36}$$

Section 5—Calculation of Acoustic Radiation Pressure on a Sphere

In general the coefficients A_n of the incident radiation field (29) will be complex, so that we write

$$A_n = |A_n|\, e^{i(\omega t + \alpha_n)}. \tag{37}$$

It is also convenient to write

$$F_n(\alpha) + i\,G_n(\alpha) = H_n(\alpha)\, e^{i\varepsilon_n},$$

where

$$H_n(\alpha) = \{F_n^2(\alpha) + G_n^2(\alpha)\}^{\frac{1}{2}},\ \cos\varepsilon_n = \frac{F_n(\alpha)}{H_n(\alpha)},\ \sin\varepsilon_n = \frac{G_n(\alpha)}{H_n(\alpha)}. \tag{38}$$

We are thus enabled to write (36) in the form

$$\phi = \cos\omega t\, \Sigma\, R_n P_n(\mu) + \sin\omega t\, \Sigma\, S_n P_n(\mu), \tag{39}$$

where

$$R_n = \frac{|A_n|}{H_n(\alpha)}\frac{\cos(\alpha_n + \varepsilon_n)}{\alpha^{n+1}},\quad S_n = -\frac{|A_n|}{H_n(\alpha)}\frac{\sin(\alpha_n + \varepsilon_n)}{\alpha^{n+1}}. \tag{40}$$

Since ϕ is the velocity potential referred to, the moving centre of the sphere as origin, it follows from (17), (18), and (39) that the first-order pressure variation δp_1 at the surface of the sphere, is given by

$$\delta p_1 = \rho_0 c\kappa\{\cos\omega t\ \Sigma\, S_n P_n(\mu) - \sin\omega t\ \Sigma\, R_n P_n(\mu)\}. \tag{41}$$

Evidently the time average of this expression vanishes, so that δp_1 contributes nothing to the mean acoustic pressure.

If we denote by P_ϕ the contribution of the term $\frac{1}{2}\rho_0\dot{\phi}^2/c^2$ in (17) to the integrated component over the sphere of the z-component of the pressure variation, we have

$$P_\phi = -\frac{\pi a^2\rho_0}{c^2}\int_0^{\pi}\dot{\phi}^2\sin\theta\cos\theta\, d\theta = -\frac{\pi a^2\rho_0}{c^2}\int_{-1}^{+1}\dot{\phi}^2\mu\, d\mu. \tag{42}$$

Substituting from (39) we find on taking the time average,

$$\overline{\mathrm{P}}_\phi = -\tfrac{1}{2}\pi\alpha^2\rho_0 \int_{-1}^{1} [\{\Sigma\, \mathrm{R}_n \mathrm{P}_n(\mu)\}^2 + \{\Sigma\, \mathrm{S}_n \mathrm{P}_n(\mu)\}^2]\, \mu\, d\mu. \tag{43}$$

It is easily proved that if $m \geqslant n$,

$$\int_{-1}^{1} \mu \mathrm{P}_n(\mu)\, \mathrm{P}_m(\mu)\, d\mu = \frac{2(n+1)}{(2n+1)(2n+3)} \qquad \text{for } m = n+1$$
$$= 0 \qquad \text{for } m > n+1.$$

As a result of using this theorem, we obtain from (43),

$$\overline{\mathrm{P}}_\phi = -2\pi\alpha^2\rho_0 \sum_{n=0}^{\infty} \frac{n+1}{(2n+1)(2n+3)} (\mathrm{R}_n \mathrm{R}_{n+1} + \mathrm{S}_n \mathrm{S}_{n+1}). \tag{44}$$

Similarly, we denote by P_q the contribution of the term $-\frac{1}{2}\rho_0 q^2$ in (17) to the integrated component over the sphere of the z-component of the pressure variation.
Then we have

$$\mathrm{P}_q = \pi a^2 \rho_0 \int_0^\pi q^2 \cos\theta \sin\theta\, d\theta = \pi a^2 \rho_0 \int_{-1}^{1} q^2 \mu\, d\mu. \tag{45}$$

At the surface of the sphere we have, according to (20),

$$\partial\phi/\partial r = -\dot{\zeta}\cos\theta,$$

so that (44) becomes

$$\mathrm{P}_q = \pi a^2 \rho_0 \int_{-1}^{1} \left\{\dot{\zeta}^2 \mu^2 + \frac{1}{a^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2\right\} \mu\, d\mu.$$

The first term of the integral evidently vanishes, and we are left with

$$\mathrm{P}_q = \pi\rho_0 \int_{-1}^{1} \left(\frac{\partial\phi}{\partial\mu}\right)^2 (1-\mu^2)\, \mu\, d\mu. \tag{46}$$

On introducing the value of $\partial\phi/\partial\mu$ from (39), we find, after taking the time average, that

$$\overline{\mathrm{P}}_q = \tfrac{1}{2}\pi\rho_0 \int_{-1}^{1} [\{\Sigma \mathrm{R}_n \mathrm{P}'_n(\mu)\}^2 + \{\Sigma \mathrm{S}_n \mathrm{P}'_n(\mu)\}^2](1-\mu^2)\, \mu\, d\mu. \tag{47}$$

It is easily proved that if $m \geqslant n$,

$$\int_{-1}^{1} \mathrm{P}'_n(\mu)\, \mathrm{P}'_m(\mu)(1-\mu^2)\, \mu\, d\mu = \frac{2n(n+1)(n+2)}{(2n+1)(2n+3)} \quad \text{if } m = n.$$
$$= 0 \qquad \text{if } m > n.$$

Applying this result to (47), we obtain

$$\overline{\mathrm{P}}_q = 2\pi\rho_0 \sum_0^\infty \frac{n(n+1)(n+2)}{(2n+1)(2n+3)} (\mathrm{R}_n\mathrm{R}_{n+1} + \mathrm{S}_n\mathrm{S}_{n+1}). \tag{48}$$

Finally, we denote by P_ζ the contribution to the z-component of δp arising from the motion of the origin according to (17) and (18). Remembering that for $r = a$, $(\partial\phi/\partial r) = -\dot{\zeta}\cos\theta$, we have,

$$\mathrm{P}_\zeta = -2\pi\rho_0 a^2\dot{\zeta}\int_0^\pi \left(\dot{\zeta}\cos^2\theta + \frac{\sin\theta}{a}\frac{\partial\phi}{\partial\theta}\right)\sin\theta\cos\theta\, d\theta.$$

The first term vanishes, and on integrating the second by parts we find,

$$\mathrm{P}_\zeta = 2\pi\rho_0 a\dot{\zeta}\int_{-1}^{1} 2\mathrm{P}_2(\mu)\,\phi\, d\mu. \tag{49}$$

On substituting for ϕ from (39), it is evident that only those terms survive for which $n = 2$. There results,

$$\mathrm{P}_\zeta = 2\pi\rho_0 a\dot{\zeta}\,\frac{4}{5}\{\mathrm{R}_2\cos\omega t + \mathrm{S}_2\sin\omega t\}. \tag{50}$$

From (35) and (40), we may write

$$\dot{\zeta} = -\kappa\frac{\rho_0}{\rho_1}\frac{1}{\alpha}\{\mathrm{R}_1\cos\omega t + \mathrm{S}_1\sin\omega t\}. \tag{51}$$

On substituting in (50), and taking the time average, we obtain

$$\overline{\mathrm{P}}_\zeta = -2\pi\rho_0\,.\,\frac{2}{5}\frac{\rho_0}{\rho_1}(\mathrm{R}_1\mathrm{R}_2 + \mathrm{S}_1\mathrm{S}_2). \tag{52}$$

The total mean pressure is given by

$$\overline{\mathrm{P}} = \overline{\mathrm{P}}_\phi + \overline{\mathrm{P}}_q + \overline{\mathrm{P}}_\zeta.$$

The sum of the three contributions given by (44), (48), and (52) combine to give the series

$$\overline{\mathrm{P}} = -2\pi\rho_0\Bigg[\frac{1}{1\,.\,3}(\mathrm{R}_0\mathrm{R}_1+\mathrm{S}_0\mathrm{S}_1)\,\alpha^2 + \frac{2}{3\,.\,5}(\mathrm{R}_1\mathrm{R}_2+\mathrm{S}_1\mathrm{S}_2)\left\{\alpha^2 - 3\left(1-\frac{\rho_0}{\rho_1}\right)\right\}$$
$$+ \sum_{n=2}^\infty \frac{n+1}{(2n+1)(2n+3)}(\mathrm{R}_n\mathrm{R}_{n+1} + \mathrm{S}_n\mathrm{S}_{n+1})\{\alpha^2 - n(n+2)\}\Bigg]. \tag{53}$$

Section 6—On the Numerical Computation of Coefficients

According to (40), we easily deduce

$$R_nR_{n+1} + S_nS_{n+1}$$
$$= \frac{|A_n| \cdot |A_{n+1}|}{H_n\, H_{n+1}\, \alpha^{2n+3}} \{\cos(\alpha_{n+1} - \alpha_n) \cdot \cos(\varepsilon_{n+1} - \varepsilon_n)$$
$$- \sin(\alpha_{n+1} - \alpha_n) \cdot \sin(\varepsilon_{n+1} - \varepsilon_n)\}.$$

We find from (38),

$$\cos(\varepsilon_{n+1} - \varepsilon_n) = \frac{F_{n+1}\, F_n + G_{n+1}\, G_n}{H_{n+1}\, H_n}, \quad \sin(\varepsilon_{n+1} - \varepsilon_n) = \frac{G_{n+1}\, F_n - F_{n+1}\, G_n}{H_n\, H_{n+1}}.$$

We thus have, for purposes of computation,

$$R_n\, R_{n+1} + S_nS_{n+1} = \frac{|A_n| \cdot |A_{n+1}|}{H_n{}^2\, H_{n+1}{}^2\, \alpha^{2n+3}}$$
$$\times \{\cos(\alpha_{n+1} - \alpha_n)(F_{n+1}F_n + G_{n+1}\, G_n) + \sin(\alpha_{n+1} - \alpha_n)(F_{n+1}G_n - G_{n+1}F_n)\}. \tag{54}$$

From (33) and (34),

$$F_{n+1}\, G_n - G_{n+1}\, F_n = \{\alpha^2 - n(n+2)\}/\alpha^{2n+3} \qquad (n \neq 1), \tag{55}$$

from which we easily derive

$$(F_{n+1}\, F_n + G_{n+1}\, G_n)^2 + \frac{\{\alpha^2 - n(n+2)\}^2}{\alpha^{4n+6}} = H_{n+1}{}^2 H_n{}^2 \quad (n \neq 1). \tag{56}$$

In particular, for $n = 1$, we have from (34)

$$F_2G_1 - F_1G_2 = \{\alpha^2 - 3(1 - \rho_0/\rho_1)\}/\alpha^5,$$

so that

$$(F_2F_1 + G_2G_1)^2 + \{\alpha^2 - 3(1 - \rho_0/\rho_1)\}^2/\alpha^{10} = H_2{}^2H_1{}^2. \tag{57}$$

There now remains the problem of computing $H_n{}^2 = F_n{}^2 + G_n{}^2$. On making use of (30),

$$F_n{}^2 + G_n{}^2 = \alpha^4(\phi_{n+1}{}^2 + \psi_{n+1}{}^2)$$
$$- 2\alpha^2 n(\phi_{n+1}\phi_n + \psi_{n+1}\psi_n) + n^2(\phi_n{}^2 + \psi_n{}^2). \tag{58}$$

Since $-\alpha\phi_{n+1} = \phi'_n$ and $-\alpha\psi_{n+1} = \psi'_n$, we have

$$-\alpha(\phi_{n+1}\phi_n + \psi_{n+1}\psi_n) = \phi_n\phi'_n + \psi_n\psi'_n = \frac{1}{2}\frac{d}{d\alpha}(\phi_n{}^2 + \psi_n{}^2),$$

Remembering that $f_n = \phi_n - i\psi_n$ we write $\phi_n^2 + \psi_n^2 = |f_n|^2$, and if we notice that

$$\frac{1}{\alpha^{n-1}}\frac{d}{d\alpha}\{\alpha^n\,|f_n|^2\} = \alpha\frac{d}{d\alpha}|f_n|^2 + n\,|f_n|^2,$$

we easily find from (58),

$$G_n^2 + F_n^2 = \alpha^4\,|f_{n+1}|^2 + \frac{n}{\alpha^{n-1}}\frac{d}{d\alpha}\{\alpha^n|f_n|^2\}.$$

In terms of Bateman's function $\zeta_n(\alpha)$, we have*

$$f_n(\alpha) = -i\alpha^{-n-1}\,\zeta_n(\alpha),$$

so that

$$G_n^2 + F_n^2 = \frac{|\zeta_{n+1}|^2}{\alpha^{2n}} + \frac{n}{\alpha^{n-1}}\frac{d}{d\alpha}\left\{\frac{|\zeta_n|^2}{\alpha^{n+2}}\right\}. \tag{59}$$

Bateman† gives for $|\zeta_n(\alpha)|^2$ the following finite series,

$$|\zeta_n(\alpha)|^2 = 1 + \tfrac{1}{2}n(n+1)\,.\,\frac{1}{\alpha^2} + \tfrac{1}{2}\,.\,\tfrac{3}{4}(n-1)\,n\,(n+1)(n+2)\frac{1}{\alpha^4} + \dots$$
$$+ \frac{1\,.\,3\,\dots\,(2n-1)}{2\,.\,4\,\dots\,2n}\,.\,(2n)\,!\,\frac{1}{\alpha^{2n}}, \tag{60}$$

or

$$|\zeta_n(\alpha)|^2 = \sum_{s=0}^{n}\left(\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2s-1}{2s}\right)\frac{(n+s)\,!}{(n-s)\,!}\frac{1}{\alpha^{2s}}.$$

By means of (59) it is thus possible to express $H_n^2 = F_n^2 + G_n^2$ as polynomials in $1/\alpha^2$. Remembering that according to (31) the case $n = 1$ is exceptional, we easily find

$$|\zeta_1|^2 = 1 + \frac{1}{\alpha^2},\quad |\zeta_2|^2 = 1 + \frac{3}{\alpha^2} + \frac{9}{\alpha^4},\quad |\zeta_3|^2 = 1 + \frac{6}{\alpha^2} + \frac{45}{\alpha^4} + \frac{225}{\alpha^6},\ \text{etc.,}$$

and, in consequence, using (59),

$$\left.\begin{aligned}
H_0^2 &= \frac{1}{\alpha^2}(1+\alpha^2)\\
H_1^2 &= \frac{4}{\alpha^6}\left\{\left(1+\frac{\rho_0}{2\rho_1}\right)^2 + \frac{1}{2}\alpha^2\left(1+\frac{\rho_0}{2\rho_1}\right)\frac{\rho_0}{\rho_1} + \tfrac{1}{4}\alpha^4\right\}\\
H_2^2 &= \frac{81}{\alpha^{10}}\left\{1 + \frac{1}{9}\alpha^2 - \frac{2}{81}\alpha^4 + \frac{1}{81}\alpha^6\right\}\\
H_3^2 &= \frac{16\times 225}{\alpha^{14}}\left\{1 + \frac{1}{10}\alpha^2 + \dots\right\}\\
H_n^2 &= \frac{\{1\,.\,3\,.\,5\,\dots\,(2n-1)\}^2}{\alpha^{4n+2}}\left\{1 + \frac{n-1}{(n+1)(2n-1)}\alpha^2 + \dots\right\}
\end{aligned}\right\} \tag{61}$$

* Footnote to Section 4.

† "Partial Differential Equations," § 6.51, p. 387.

With these formulæ it is found that in plane waves two terms of the series (53) are accurate to better than 1/10% for $\alpha = \kappa a = 2\pi a/\lambda = 1$.

For large values of α exceeding 3 or 4, H_n^2 may be computed from Bessel function tables* according to the formulæ

$$\left.\begin{aligned} H_n^2 &= \tfrac{1}{2}\pi \frac{1}{\alpha^{2n+1}} \{n^2 (J_{n+\frac{1}{2}}^2 + J_{-n-\frac{1}{2}}^2) + 2n\alpha (J_{-n-\frac{1}{2}} J_{-n-\frac{3}{2}} - J_{n+\frac{1}{2}} J_{n+\frac{3}{2}}) \\ &\qquad + \alpha^2 (J_{n+\frac{3}{2}}^2 + J_{-n-\frac{3}{2}}^2)\}, \qquad (n \neq 1) \\ H_1^2 &= \tfrac{1}{2}\pi \frac{1}{\alpha^3} \left\{\left(1 - \frac{\rho_0}{\rho_1}\right)^2 (J_{\frac{3}{2}}^2 + J_{-\frac{3}{2}}^2) + 2\alpha \left(1 - \frac{\rho_0}{\rho_1}\right) (J_{-\frac{3}{2}} J_{-\frac{5}{2}} - J_{\frac{3}{2}} J_{\frac{5}{2}}) \right. \\ &\qquad \left. + \alpha^2 (J_{\frac{5}{2}}^2 + J_{-\frac{5}{2}}^2)\right\} \end{aligned}\right\}. \qquad (62)$$

Section 7—Radiation Pressure on a Sphere in a Plane Progressive Wave

To express a plane wave in terms of spherical wave functions, we use an expansion due to Bauer,† and write for the velocity potential of the incident radiation field.

$$\phi_i = Ae^{-i\kappa z} = A \sum_{n=0}^{\infty} (2n + 1) (- i)^n \psi_n (\kappa r) . (\kappa r)^n P_n (\mu). \qquad (63)$$

From (37) it follows that

$$A_n = (2n + 1) |A| e^{i(\omega t - \frac{1}{2}n\pi)}, \qquad \text{so that} \qquad \alpha_n = - \tfrac{1}{2}n\pi,$$

and

$$|A_n| = (2n + 1) |A|.$$

Referring to (54), we have $\alpha_{n+1} - \alpha_n = - \frac{1}{2}\pi$, and in consequence,

$$R_n R_{n+1} + S_n S_{n+1} = - \frac{|A|^2 (2n + 1) (2n + 3)}{H_n^2 H_{n+1}^2 \alpha^{2n+3}} \{F_{n+1} G_n - G_{n+1} F_n\}. \qquad (64)$$

If we use (55), we find that the general formulæ (53) for the mean pressure gives,

$$\overline{P} = 2\pi\rho_0 \frac{|A|^2}{\alpha^2} \left[\frac{1}{H_0^2 H_1^2} + \frac{2}{H_1^2 H_2^2} \frac{\{\alpha^2 - 3 (1 - \rho_0/\rho_1)\}^2}{\alpha^8} + \sum_{n=2}^{\infty} \frac{(n + 1)}{H_n^2 H_{n+1}^2} \frac{\{\alpha^2 - n (n + 2)\}^2}{\alpha^{4n+4}}\right]. \qquad (65)$$

* "Report Brit. Assoc.," 1925, p. 221.

† Watson, 'Theory of Bessel Functions,' 4.82 (1922); also Lamb, 'Hydrodynamics,' Section 296.

It is evident from this expression that the radiation pressure on a sphere in a progressive wave is always positive, *i.e.*, in the direction of wave propagation.

Radiation Pressure on Small Spheres ($\alpha = \kappa a \leqslant 1$)

When the circumference of the sphere is very small compared to the wave-length, we find on using (61) that (63) yields

$$\overline{\mathrm{P}} = 2\pi\rho_0 |\mathrm{A}|^2\alpha^6 \frac{\{1 + \frac{2}{9}(1 - \rho_0/\rho_1)^2\}}{(2 + \rho_0/\rho_1)^2} + \text{terms in } \alpha^8 \text{ and higher powers.} \quad (66)$$

It is interesting to note that the relative density factor

$$\mathrm{F}(\rho_0/\rho_1) = \frac{1 + \frac{2}{9}(1 - \rho_0/\rho_1)^2}{(2 + \rho_0/\rho_1)^2}, \quad (67)$$

is always finite and positive in the interval $0 < \rho_0/\rho_1 < \infty$, and has a minimum at $\rho_0/\rho_1 = 3$. Its value is tabulated below :—

ρ_0/ρ_1	0	1	3	∞
$\mathrm{F}(\rho_0/\rho_1)$	0·305	0·111	0·075	0·222

The last entry is the value appropriate to a rigid bubble.

The mean total energy-density in the wave is $\overline{\mathrm{E}} = \frac{1}{2}\rho_0\kappa^2|\mathrm{A}|^2$. We may write (66) in the form

$$\frac{\overline{\mathrm{P}}}{\pi a^2} \sim 4(\kappa a)^4 \mathrm{F}(\rho_0/\rho_1) . \overline{\mathrm{E}} = 4\left(\frac{2\pi a}{\lambda}\right)^4 \mathrm{F}(\rho_0/\rho_1) . \overline{\mathrm{E}} \quad (68)$$

as the results of observation are usually expressed in this manner.

Radiation pressure on a sphere for which $\alpha = \kappa a = 1$

We easily find that the third term of (65) is of the order $\dfrac{49}{89 \times 1200}$. Neglecting this and higher terms, we find

$$\overline{\mathrm{P}} \sim \pi\rho_0 |\mathrm{A}|^2 \frac{1}{89} \frac{95 - 48(\rho_0/\rho_1) + 36(\rho_0/\rho_1)^2}{5 + 6(\rho_0/\rho_1) + 2(\rho_0/\rho_1)^2}. \quad 69)$$

In the case of sound waves in water, we take $\rho_0/\rho_1 = 1$. We then find that

$$\frac{\overline{\mathrm{P}}}{\pi a^2} \sim \frac{166}{13 \times 89} \overline{\mathrm{E}} = 0 \cdot 143\, \overline{\mathrm{E}}. \quad (70)$$

The mean pressure in this case is very much greater than that given by (68). It is found, however, that when such a sphere is made up into a torsion balance of optimum sensitivity* of 10 seconds period, it requires waves transmitting 49 erg/cm² (107 decibels), on a 2 cm wave-length in water to give a deflection $\theta = 1/(2000)$, *i.e.*, 1 mm on a scale at 1 metre. We conclude that in a progressive plane wave in water the radiation pressure on a sphere is too small to be observed except when the intensity exceeds a level of 100 decibels.

Section 8—Radiation Pressure on a Sphere in a Plane Stationary Wave

If the centre of the sphere be at distance h from a fixed plane of reference, the incident velocity potential referred to the mean position of the centre as origin may be written

$$\phi_i = \tfrac{1}{2}\mathrm{A}\,\{e^{-i\kappa(z+h)} + e^{i\kappa\,(z+h)}\}, \quad \text{where} \quad \mathrm{A} = |\mathrm{A}|\, e^{i\omega t}. \tag{71}$$

According to (63) we may write

$$e^{i\kappa(z+h)} = \sum_0^\infty (2n+1)\, e^{i(\kappa h+\frac{1}{2}n\pi)}\, \psi_n(\kappa r).(\kappa r)^n\, \mathrm{P}_n(\mu),$$

and have, on changing the sign of i and adding,

$$\phi_i = \mathrm{A}\sum_0^\infty (2n+1)\cos(\kappa h + \tfrac{1}{2}n\pi)\quad \psi_n(\kappa r).(\kappa r)^n \quad \mathrm{P}_n(\mu). \tag{72}$$

From (29) and (37) it follows that $|\mathrm{A}_n| = |\mathrm{A}|\,(2n+1)\cos(\kappa h + \frac{1}{2}n\pi)$ and $\alpha_n = 0$.

Equations (40) give

$$\mathrm{R}_n = \frac{|\mathrm{A}|}{\mathrm{H}_n\,\alpha^{n+1}}(2n+1)\cos(\kappa h + \tfrac{1}{2}n\pi)\cos\varepsilon_n,$$

$$\mathrm{S}_n = -\frac{|\mathrm{A}|}{\mathrm{H}_n\,\alpha^{n+1}}(2n+1)\cos(\kappa h + \tfrac{1}{2}n\pi)\sin\varepsilon_n,$$

from which it follows that

$$\mathrm{R}_n\mathrm{R}_{n+1}+\mathrm{S}_n\mathrm{S}_{n+1} = \frac{(2n+1)(2n+3)}{2\,\alpha^{2n+3}}\,|\mathrm{A}|^2\,(-1)^{n+1}\sin 2\kappa h\,\frac{(\mathrm{F}_{n+1}\mathrm{F}_n+\mathrm{G}_{n+1}\mathrm{G}_n)}{\mathrm{H}_n{}^2\,\mathrm{H}_{n+1}{}^2}. \tag{73}$$

* Theoretical considerations on the design of torsion balances of optimum sensitivity using radiation pressures on spheres and circular discs is dealt with in another paper.

Finally, the general formula (53) gives for the mean pressure

$$\overline{\mathrm{P}} = \pi\rho_0\,|\mathrm{A}|^2 \sin 2\kappa h\left[\frac{1}{\alpha^3}\frac{(\mathrm{F_0F_1}+\mathrm{G_0G_1})}{\mathrm{H_0}^2\,\mathrm{H_1}^2}\,\alpha^2 - \frac{2}{\alpha^5}\frac{(\mathrm{F_1F_2}+\mathrm{G_1G_2})}{\mathrm{H_1}^2\,\mathrm{H_2}^2}\{\alpha^2-3\,(1-\rho_0/\rho_1)\}\right.$$
$$\left. + \sum_{n=2}^{\infty}(-1)^n\,\frac{(n+1)}{\alpha^{2n+3}}\,\frac{(\mathrm{F}_{n+1}\mathrm{F}_n+\mathrm{G}_{n+1}\mathrm{G}_n)}{\mathrm{H}_n{}^2\,\mathrm{H}_{n+1}{}^2}\{\alpha^2-n\,(n+2)\}\right]. \quad (74)$$

Radiation Pressure on Small Spheres ($\alpha = ka \ll 1$)

It is obvious from (23) and (24) that $\mathrm{G}_{n+1}\mathrm{G}_n \ll \mathrm{F}_{n+1}\mathrm{F}_n$. Using (30), we find when $\alpha \ll 1$ the approximate expansions,

$$\mathrm{F_0} \sim \frac{1}{\alpha},\quad \mathrm{F_1} \sim \frac{1}{\alpha^3}\,(2+\rho_0/\rho_1),\quad \mathrm{F_2} \sim \frac{9}{\alpha^5},\quad \mathrm{F_3} \sim \frac{60}{\alpha^7},\ \text{etc.},$$

and, in general, $\mathrm{F}_n \sim 1\,.\,3\,.\,\ldots\,(2n-1)\,(n+1)/\alpha^{2n+1}$.

In the present circumstances the first three terms of (74) give

$$\overline{\mathrm{P}} \sim \pi\rho_0|\mathrm{A}|^2 \sin 2\kappa h\left[\frac{1}{\alpha}\cdot\frac{1}{\mathrm{F_0F_1}} - \frac{2}{\alpha^5}\frac{\{\alpha^2-3\,(1-\rho_0/\rho_1)\}}{\mathrm{F_1F_2}} + \frac{3\,(\alpha^2-8)}{\alpha^7\,\mathrm{F_2F_3}} - \ldots\right], \quad (75)$$

from which we derive the approximate expression

$$\overline{\mathrm{P}} = \pi\rho_0|\mathrm{A}|^2 \sin 2\kappa h\,\alpha^3\frac{\{1+\frac{2}{3}\,(1-\rho_0/\rho_1)\}}{2+\rho_0/\rho_1} + \text{terms in } \alpha^4 \text{ and higher terms.} \quad (76)$$

It is important to notice that the leading term is of the order $(\kappa a)^3$, so that the mean radiation pressure on a small sphere due to stationary waves is of a much greater order of magnitude than that exerted by progressive waves, the maximum amplitude being the same in both cases.

According to (71), the velocity potential of the field is

$$\phi_i = |\mathrm{A}|\cos \kappa h \cos \omega t,$$

so that the particle velocity is $\dot{\xi} = \mathrm{A}\kappa \sin \kappa h \cos \omega t$.
The nodes ($\dot{\xi} = 0$) are at $h = 0,\ \pm\,\pi/\kappa,\ \pm\,2\pi/\kappa,\ \ldots\ \pm\,s\pi/\kappa$, and the loops are situated at $h = \pm\,\frac{1}{2}\pi/\kappa,\ \pm\,\frac{3}{2}\pi/\kappa,\ \ldots\ \pm\,(s+\frac{1}{2})\,\pi/\kappa$.

It will readily be seen that for a relatively dense sphere ($\rho_0/\rho_1 < 2\cdot5$), the mean radiation pressure is such as to urge it away from the nodes towards the loops. On the other hand, a relatively light sphere ($\rho_0/\rho_1 > 2\cdot5$) is urged away from the loops towards the nodes. The question is examined in greater detail in the following section.

As in §7, the mean total energy-density in a stationary wave field is $\bar{E} = \frac{1}{2}\rho_0\kappa^2|A|^2$ at all points of the field. We may thus write (76) in the form,

$$\frac{\bar{P}}{\pi a^2} \sim 2\,(\kappa a) \sin 2\kappa h\, F\,(\rho_0/\rho_1)\,.\,\bar{E}, \tag{77}$$

where, in this case, the relative density factor is

$$F\,(\rho_0/\rho_1) = \frac{1 + \frac{2}{3}\,(1 - \rho_0/\rho_1)}{2 + \rho_0/\rho_1}. \tag{78}$$

It is interesting to notice that (77), as regards its linear dependence on κa follows the early trend of observations by Boyle and Lehmann* relating to pressures on circular discs of lead placed in the central beam of a supersonic piezo-electric oscillator, where the radiation field is of quasi-stationary character, and where the stationary component of the field is the dominant one.

It is possible to use (76) or (77) as the basis for the design of " spherical torsion balances " of optimum sensitivity for use in liquid or gaseous media, the detailed consideration of which must, however, be left to a future paper.

It is now evident on comparing (66) and (76) that the mean radiation pressure on a small sphere does not depend on the local specifications of the field, such as the total mean energy-density. The pressure depends, in fact, on the nature of the field as a whole as related to the mode of generation of the sound waves. In other than the simple types of field discussed in this paper, the computation of the components of radiation pressure is extremely difficult, especially if the sphere is free to oscillate as the effect of the first-order pressure. Its centre will, in general, move in a small three-dimensional orbit, with the result that the relative density factor will be different for the three components of the mean radiation pressure. The procedure of the present paper might, without difficulty, be generalized to give the mean radiation pressure in a complex progressive or stationary plane wave, or a complex wave made up of progressive and stationary components. The final result is, however, of no very great interest, as the main features of pressure effects are well illustrated by the simple problems considered in Sections 7 and 8.

* 'Can. J. Res.,' vol. 3, p. 505 (1930). The writer has, in fact, obtained for the radiation pressure on small thin circular discs in progressive and stationary plane waves formulæ similar to (66) and (76) except for the relative density factor which takes different forms. The investigation will be published in a later paper.

SECTION 9—ON THE MOTION OF SMALL SPHERES IN A PLANE STATIONARY RADIATION FIELD

If the sphere is free to move under the influence of the mean radiation pressure, we must add to the velocity potential ϕ the term

$$\phi_s = \dot{h}a^3 \cos \theta / r^2$$

appropriate to the mean velocity $\dot{h}$ and satisfying the boundary condition $-(\partial \phi_s/\partial r)_{\kappa=a} = \dot{h} \cos \theta$. The pressure arising from this mean velocity when integrated over the sphere is easily found to be

$$\overline{\mathrm{P}}_s = -\mathrm{M}'\ddot{h}, \quad \text{where} \quad \mathrm{M}' = \tfrac{2}{3}\pi a^3 \rho_0. \tag{79}$$

The equation of motion of the sphere under the influence of the mean radiation pressure is thus seen to be

$$\mathrm{M}\ddot{h} = \overline{\mathrm{P}} + \overline{\mathrm{P}}_s. \tag{80}$$

On using (76) for the value of $\overline{\mathrm{P}}$ in a plane stationary wave and (79) for the value of $\overline{\mathrm{P}}_s$ due to the mean motion, the equation (80) may be written

$$(\mathrm{M} + \mathrm{M}')\ddot{h} = \pi\rho_0|\mathrm{A}|^2 \sin 2\kappa h \, (\kappa a)^3 \frac{\{1 + \frac{2}{3}(1 - \rho_0/\rho_1)\}}{2 + \rho_0/\rho_1}. \tag{81}$$

If we introduce the new variable θ given by $2\kappa h = \pi - \theta$, equation (81) takes the form

$$\ddot{\theta} + n^2 \sin \theta = 0, \tag{82}$$

where

$$n^2 = 3|\mathrm{A}|^2 \kappa^4 \frac{\rho_0}{\rho_1} \frac{\{1 + \frac{2}{3}(1 - \rho_0/\rho_1)\}}{(2 + \rho_0/\rho_1)^2}. \tag{83}$$

From the last two equations it is easily seen that the position of stable equilibrium depends on the sign of the relative density factor. We have the two cases :

(1) for relatively dense spheres ($\rho_0/\rho_1 < 2 \cdot 5$), the position of stable equilibrium is at $\theta = 0$, *i.e.*, at $h = \pm \frac{1}{2}\pi/\kappa$ and, in general, at $h = \pm (\mathrm{S} + \frac{1}{2}) \pi/\kappa$, *i.e.*, at the loops of the standing waves.

(2) for relatively light spheres ($\rho_0/\rho_1 > 2 \cdot 5$), the position of stable equilibrium is at $\theta = \pi$, *i.e.*, at $h = 0$, and in general, at $h = \pm s\pi/\kappa$, *i.e.*, at the nodes of the standing waves.

It is interesting to notice that the motion as governed by equations (82) and (83) is *independent of the radius of the sphere*, and is of the same nature as the oscillations of a simple pendulum. The time of a complete oscillation from a position of rest at $h = h_0$ is easily seen to be

$$\mathrm{T} = \frac{2\pi}{n} \cdot \frac{2}{\pi} \mathrm{K}(k), \tag{84}$$

where K (k) is the complete elliptic integral to modulus $k = \cos \kappa h_0$.

In terms of the velocity amplitude of the sound waves, $|\dot{\xi}| = \kappa|\mathrm{A}|$, the wave-length given by $\kappa = 2\pi/\lambda$, and the relative density factor $f(\rho_0/\rho_1)$ given by

$$f(\rho_0/\rho_1) = \left\{\frac{(2 + \rho_0/\rho_1)^2}{(\rho_0/\rho_1)(5 - 2\rho_0/\rho_1)}\right\}^{\frac{1}{2}}, \tag{85}$$

while we have for the time of oscillation of small spheres of all radii such that $\kappa a \ll 1$, the convenient formula,

$$\mathrm{T} = \frac{\lambda}{|\dot{\xi}|} f(\rho_0/\rho_1) \cdot \frac{2}{\pi} \mathrm{K}(k), \quad \text{where} \quad k = \cos(2\pi h_0/\lambda). \tag{86}$$

For relatively dense spheres $(\rho_0/\rho_1 > 2 \cdot 5)$, it is evident from (85) that the factor $f(\rho_0/\rho_1)$ is infinite at $\rho_0/\rho_1 = 0$, and at $\rho_0/\rho_1 = 2 \cdot 5$. In the interval it has a minimum at $\rho_0/\rho_1 = 10/13$, for which value $f(\rho_0/\rho_1) = 1 \cdot 67$ and the time of oscillation is a minimum. In these circumstances we have,

$$\mathrm{T}_{\min} = 1 \cdot 67 \, (\lambda/|\dot{\xi}|) \frac{2}{\pi} \mathrm{K}(k), \tag{87}$$

the oscillations in this case being across a loop of the standing wave system.

The amplitude factor $(2/\pi)$ K (k) varies slowly with the distance of the initial position from the loop as shown in the following table :—

Distance from loop	0	$\frac{1}{4}\lambda$	$\frac{1}{3}\lambda$	$\frac{5}{12}\lambda$	$\frac{1}{2}\lambda$ (node)	
$\frac{2}{\pi}$ K (k)	1	1·18	1·37	1·76	∞	(88)

The formula (87) has a bearing on the well-known phenomena exhibited when small particles are introduced in a high-frequency stationary wave field. For instance, in water, a velocity amplitude $|\dot{\zeta}| = 1 \cdot 53$ cm sec at wave-length $3 \cdot 77$ cm corresponds to a very moderate radiation density of $0 \cdot 0175$ watts/cm^2, emitted by a piezo-electric oscillator of

30 cm radius surrounded by an infinite rigid flange and radiating 50 watts.* Near the disc the radiation-field is very nearly of a stationary character. According to the results of the present section sharply defined " dust striations " are best observed with spherical particles of optimum density $1 \cdot 3$, since the time of oscillation across a loop is then a minimum.

With the above values we find

$$T_{min} = 4 \cdot 1 \times \frac{2}{\pi} K(k) \text{ sec.} \tag{89}$$

A reference to (88) tells us that if the particles are initially equally distributed, almost all of them with the exception of those near the nodes will be swept towards the nearest loop in a time of the order $\frac{1}{4}T_{min}$, *i.e.*, in a time interval of 1 or 2 seconds. It is thus probable that radiation pressure alone is the explanation of the remarkable striations observed by Boyle† and his co-workers in the radiation field of a supersonic piezo-electric oscillator. The sharpness of definition of the striations is no doubt due to the fact that the motion is independent of particle size.

When the particles are very small, the influence of viscosity on the mean motion cannot be ignored, we therefore discuss this aspect of the problem in greater detail in a later section.

Section 10—On the Influence of Radiation Pressure on Dust Striations in Resonance Tubes

It appears probable to the writer that radiation pressure plays an important part in the formation of the well-known striated patterns which light particles in a resonance tube assume when standing waves are excited therein.‡ The mean features of the phenomena are described in all text-books on sound and are familiar to all students of physics. Recently, the formation of these striations have been studied under carefully controlled conditions by Andrade.§

* The properties of this type of radiation field have been worked out in detail by the author in a recent paper, (Can. J. Res., Vol. 11, p.135, August, 1934).

† Boyle, Lehmann, and Reid, 'Trans. Roy. Soc. Canada,' vol. 19, Section 3, p. 167 (1925). Boyle and Lehmann, *Ibid.*, p. 159. It is stated, (p. 161) that particles of cinder dust settle in nodal planes in a stationary radiation field. According to a private communication from Dr. Boyle the density of the dust used in these experiments was : ashes, $0 \cdot 72$; lignite, 1.1 ; bituminous coal, $1 \cdot 2$–$1 \cdot 5$. These densities are considerably less than $2 \cdot 5$, the particles are "relatively light," and as required by theory are driven by acoustic radiation pressure towards the nodes.

‡ Rayleigh, " Theory of Sound," vol. II pp. 46, 50 (1896).

§ 'Phil. Trans.,' A, vol. 230, p. 413 (1932).

Considering the problem in the light of a radiation pressure effect, we shall suppose that we have to deal with small spherical particles still sufficiently large that their inertia is the important term in the equation of motion (80) and that viscosity plays a minor role. In these circumstances, the efficacy of radiation pressure in forming striations may be judged by the time of oscillation given by (86). In air, it is obviously impossible to adjust the relative density factor to a minimum. The best that can be done is to choose a material for the particles for which ρ_0/ρ_1 is as close as possible to the optimum ratio $0\cdot755$; that is we have to use very light particles such as cork spherules, for which $\rho_1 = 0\cdot24$.

Taking $\rho_0 = 0\cdot0012$ as the density of air, $\rho_0/\rho_1 = 5 \times 10^{-3}$ and $f(\rho_0/\rho_1) = 12\cdot5$.

In his paper Andrade quotes observed displacement amplitudes of aerial vibrations of the order $|\zeta| = 10^{-2}$ cm at a frequency $f = 500$. This corresponds to a velocity amplitude $|\dot{\zeta}| = 31\cdot4$ cm/sec at a wavelength of $\lambda = 68\cdot8$ cm.

Since $\rho_0/\rho_1 < 2\cdot5$, the motion is that characteristic of relatively dense spheres which according to (86) oscillate across the loops or anti-nodes with a period

$$\mathrm{T} = 27\cdot4 \times \frac{2}{\pi}\mathrm{K}\,(k) \text{ seconds.}$$

In the light of (88) it is evident that, if the particles are initially equally distributed, they will with the exception of those near the nodes be swept towards the nearest loop or anti-node in a time of the order $\frac{1}{4}$T, *i.e.*, in 7–13 secs. At the higher frequencies and greater acoustic amplitudes at which the experiment is usually carried out this time of formation of the striations will be very much less.

Reference to the illustrations in Andrade's paper show that when precautions are taken to generate pure waves of moderate amplitudes, and to minimize the transmission of waves by the walls of the resonance tube, small particles tend to gather at the loops of the standing waves in the form of sharply defined *antinodal discs*. Circulation of air in the form of vortex patterns undoubtedly play an important part in the phenomenon, the particles forming the antinodal discs must be held against gravity by a radial vortex circulation, but their extreme sharpness of definition is apparently due to the effect of radiation pressure. The mutual effect of particles in close proximity is such as to further enhance this sharpness of definition as explained by Konig.*

* Rayleigh, "Theory of Sound," p. 46 (1896).

As usually performed, Kundt's experiment gives rise to prominent striations at the nodes of the stationary wave system. If the tube walls are thin, stationary fluxual waves in the tube are set up, and these again give rise to an associated wave-system in the air. A node in the original longitudinal sound waves will correspond to a node in the fluxual waves of the tube which consist of stationary swellings and contractions. The velocity potential corresponding to the aerial waves thus set up reveals the existence of longitudinal amplitudes of motion having maximum values or loops across the nodes close to the wall and diminishing rapidly with distance from it. In this radiation field, small spherical particles near the walls will be driven by radiation pressure to the loops of these secondary transverse waves, which for certain wall constants may attain high amplitudes of a resonance type. It is thus possible to account for the formation of the "nodal wall striations" which often make their appearance.

To explain the inter-nodal fine structure striations frequently observed at high intensities, it must be remembered that even if the source of the sound, such as an oscillating diaphragm, has a purely harmonic motion, the resulting waves of large amplitude have a complex character, the higher harmonics being present.* These react on the walls giving rise to fluxual oscillations, which in turn generate secondary stationary sound waves in which radiation pressure may collect light particles into inter-nodal striations in the manner already explained.

It is also probable that in a similar way radiation pressure in the acoustic radiation field associated with vibrating plates may play a part in driving relatively heavy particles towards the nodes, while extremely small particles in the motion of which viscosity plays the leading part are caused to drift towards the loops by the vortex circulation engendered. A detailed mathematical investigation of the part played by radiation pressure in the formation of Chladni and Savart's sand figures† is, however, beyond the scope of the present paper.

SECTION 11—ON THE INFLUENCE OF RADIATION PRESSURE ON THE FORMATION OF DUST STRIATIONS IN STATIONARY WAVES

We consider a system of plane stationary waves whose nodal planes N, fig. 1, are vertical. A small sphere of density ρ_1 is released at a

* Lamb, "Dynamical theory of sound," Arnolds, p. 180 (1910); also King, 'Phil. Trans.,' A, vol. 218, p. 221 (1919).

† Rayleigh, "Theory of Sound," vol. 1, p. 367 (1894).

distance x_0 from the nearest anti-nodal plane L which contains the axis of z measured vertically downwards, the origin being taken in the horizontal plane in which the particle is released. The effect of viscosity is to introduce a force $\mu\dot{x}$ opposing the motion which, introduced in equation (81) gives

$$(\mathrm{M}+\mathrm{M}')\ddot{h}+\mu\dot{h}=2\pi\rho_0(\kappa a)^3 f(\rho_0/\rho_1)\sin\kappa h\cos\kappa h, \tag{90}$$

where

$$f(\rho_0/\rho_1)=\{1+\tfrac{2}{3}(1-\rho_0/\rho_1)\}/(2+\rho_0/\rho_1). \tag{91}$$

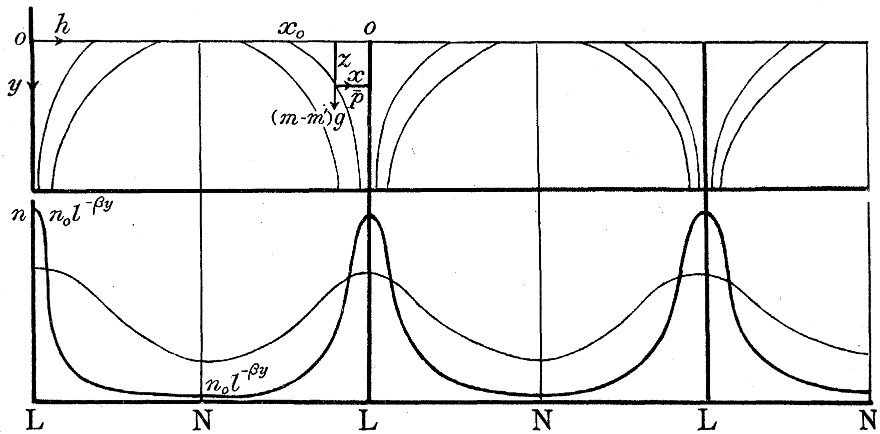

FIG. 1 (a)—Sketch of trajectories of small spheres moving under the effect of gravity, viscosity, and radiation pressure in a system of horizontal plane standing waves.
FIG. 1 (b)—Sketch of relative densities of small particles after falling through a vertical distance z in the radiation field of fig. 1 (a). The initial density distribution over the plane $z = 0$ is constant.

The position of the sphere is defined by the distance h measured from left to right from an arbitrary plane of reference. If we are dealing with relatively dense spheres ($\rho_0/\rho_1 > 2\cdot5$), the motion will take place towards the nearest loop. In terms of the co-ordinate x of fig. 1 (a) measured from right to left from the axis of z situated in the plane of loops, we write

$$h=\tfrac{1}{2}\pi/\kappa - x, \text{ and (91) becomes}$$

$$(\mathrm{M}+\mathrm{M}')\ddot{x}+\mu\dot{x}=-2\pi\rho_0|\mathrm{A}|^2(\kappa a)^3 f(\rho_0/\rho_1)\sin\kappa x\cos\kappa x, \tag{92}$$

while the equation of motion under the effect of gravity is

$$(\mathrm{M}+\mathrm{M}')\ddot{z}+\mu\dot{z}=\tfrac{4}{3}\pi a^3(\rho_1-\rho_0)g. \tag{93}$$

When the particles are sufficiently small, the effect of viscosity predominates, and we may neglect the acceleration terms in (92) and (93). In these circumstances the integrals of the equations of motion are

$$\frac{\mu}{\kappa}\log\frac{\tan\kappa x_0}{\tan\kappa x} = 2\pi\rho_0 |A|^2 (\kappa a)^3 f(\rho_0/\rho_1)\,t, \text{ and } \mu z = \tfrac{4}{3}\pi a^3(\rho_1 - \rho_0) g t, \quad (94)$$

so that on eliminating t, the equation of the trajectory is, on using (90),

$$\tan\kappa x = \tan\kappa x_0\, e^{-\beta z}, \quad (95)$$

where

$$\beta = \frac{|A|^2\kappa^4}{2g}\frac{(5 - 2\rho_0/\rho_1)\,.\,\rho_0/\rho_1}{(2+\rho_0/\rho_1)(1-\rho_0/\rho_1)}. \quad (96)$$

It is interesting to note that the trajectory is independent of particle radius and viscosity so that all trajectories from the same initial position are identical.

We now suppose the particles to be initially equally distributed in the plane $z = 0$, so that the number between x_0 and $x_0 + dx_0$ is $n_0 dx$, n_0 being constant. Then if at depth z the number between x and $x + dx$ is ndx, we must obviously have

$$ndx = n_0\, dx_0. \quad (97)$$

We accordingly find from (95) that the density distribution at depth z is given by

$$n = \frac{n_0}{\cosh\beta z - \cos 2\kappa x \sinh\beta z}. \quad (98)$$

Two such distribution curves are sketched in fig. 1 (*b*). For relatively dense particles the density distribution n has minima $n_0 e^{-\beta z}$ at the nodes and maxima $n_0 e^{\beta z}$ at the loops.

Numerical Examples

(i) Take $\rho_0/\rho_1 = 0\cdot 5$, $\lambda = 3\cdot 77$ cm, $|\dot\xi| = \kappa|A| = 1\cdot 53$ cm/sec. As in Section 9, $|\dot\xi|$ is the velocity amplitude of the approximately stationary field near the disc of a piezo-electric oscillator emitting $0\cdot 0175$ watts/cm². Supposing the particles to fall through 30 cm, we easily find from (96) that $\beta z = 0\cdot 159$, so that relative densities of the particles at this depth at the loops and nodes is $e^{\beta z} : e^{-\beta z} = 1\cdot 17 : 0\cdot 85$, which is just detectable.

(ii) Boyle's experiments were carried out at $f = 570{,}000$ for which $\lambda = 0\cdot 258$ cm. For the same velocity amplitude $|\dot\xi| = 1\cdot 53$ cm/sec corresponding to the same radiation density as in the previous example, we find for

$$z = 10 \text{ cm}, \ \beta z = 11\cdot 1, \text{ and } e^{\beta z} : e^{-\beta z} = 6\times 10^4 : 1\cdot 7\times 10^{-4},$$

while for $z = 5$ cm, $\beta z = 5\cdot 6$, and $e^{\beta z} : e^{-\beta z} = 257 : 0\cdot 0039$.

At these extremely high frequencies it is evident that radiation pressure is able to produce very sharp striations in the pattern of small relatively dense particles allowed to fall through moderate distances. A large variety of these patterns have been observed by Boyle, Lehmann, and Reid, to whose paper the reader is referred for experimental details.

It is worthy of notice that if a count or microphotometric record of particle density be made in a plane stationary wave field resulting from an initially uniform distribution allowed to fall under gravity through a distance z, βz may be roughly evaluated from (98). If the wave-length and density of the particles are known, equation (96) allows the velocity amplitude $|\dot{\xi}| = \kappa\,|\mathrm{A}|$ to be evaluated. The same object may be achieved if it is possible to photograph the trajectory of a single particle.

We may notice that relatively light spherules ($\rho_0/\rho_1 > 2\cdot 5$), such as air bubbles, tend to move towards the nodal planes. Observations on the motion of such particles do not appear to have been published, and their investigation would be of great interest.*

If we start with an initially uniform distribution of N_0 particles per unit volume, the final distribution N per unit area at depth z is easily found to be,

$$\mathrm{N} = \int_0^z \frac{dx}{dx_0}\,dz = \frac{\mathrm{N}_0}{\beta}\,\frac{\tan^{-1}(e^{\beta z}\tan\kappa x) - \kappa x}{\sin\kappa x\cos\kappa x}. \tag{99}$$

At the loops, ($\kappa x = 0$), $\mathrm{N}_\mathrm{L} = (\mathrm{N}_0/\beta)\,(e^{\beta z} - 1)$, while at the nodes $\mathrm{N}_\mathrm{N} = (\mathrm{N}_0/\beta)\,(1 - e^{-\beta z})$, and $\mathrm{N}_\mathrm{L} : \mathrm{N}_\mathrm{N} = 1 : e^{-\beta z}$. The "contrast" of particle density at the loops and nodes is thus less accentuated than when the particles are initially uniformly distributed in a plane. In the numerical example (i) above, $\mathrm{N}_\mathrm{L} : \mathrm{N}_\mathrm{N} = 1 : 0\cdot 85$, while in example (ii) we have for $z = 10$ cm, $\mathrm{N}_\mathrm{L} : \mathrm{N}_\mathrm{N} = 1 : 1\cdot 7 \times 10^{-4}$, and for $z = 5$ cm, $\mathrm{N}_\mathrm{L} : \mathrm{N}_\mathrm{N} = 1 : 3\cdot 9 \times 10^{-3}$. In the latter case the striations are as sharply accentuated as could be desired.

Section 12—On the Suspension of Small Spherical Particles against Gravity

Of some interest from the experimental point of view is the possibility of suspending small spherical particles against gravity in a vertical column of fluid in which supersonic stationary waves are generated by

* This conclusion is verified by Boyle's observations. (Boyle, Taylor and Froman, 'Trans. Roy. Soc. Canada', vol. 23, section III, p. 189 (1929)). It is distinctly stated that bubbles collect at the nodes of the stationary waves as required by theory.

a piezo-electric oscillator. If we equate $\overline{P}$ given by (76) to the buoyancy, we obtain for the velocity amplitude $|\dot{\xi}| = \kappa\,|A|$ the equation

$$|\dot{\xi}|^2 = \frac{2\lambda g}{\pi}\frac{1}{\sin 2\kappa h}\cdot\frac{(1-\rho_0/\rho_1)(2+\rho_0/\rho_1)}{(\rho_0/\rho_1)(5-2\,\rho_0/\rho_1)}. \tag{100}$$

Numerical Examples

(i) *Light particles in air*—If we consider small spherules of cork in a vertical column of air for which $\rho_1 = 0{\cdot}24$, $\rho_0 = 0{\cdot}0012$, the necessary velocity amplitude for equilibrium at a loop is given by $|\dot{\xi}|^2 = 51\ \lambda g$. Even at supersonic frequencies the velocity amplitude required is much beyond that experimentally realizable.

(ii) *Small particles in water*—It is evident that by taking $\rho_1/\rho_0 \sim 1$, equilibrium is always possible at the loops for nearly buoyant spherules.

If we take $\rho_0/\rho_1 = 0{\cdot}5$, $\lambda = 0{\cdot}258$ cm as in the example (ii) of Section 11, we find for the minimum velocity amplitude for which equilibrium is possible at a loop,

$$|\dot{\xi}|^2 = \tfrac{5}{4}\lambda g/\pi, \text{ giving } |\dot{\xi}| = 10 \text{ cm/sec},$$

a value which should be easily attainable in practice.

If, in the same circumstances, we take ρ_0/ρ_1 very large, the corresponding minimum velocity amplitude for which small rigid bubbles can be held in equilibrium at a node is $|\dot{\xi}|^2 = \lambda g/\pi$, giving $|\dot{\xi}| = 9$ cm/sec, also realizable experimentally.

These results are of interest in connection with experiments of Biquard* in which the velocity amplitude of stationary waves shows a marked exponential decrease with distance from the piezo-electric oscillator not accounted for by the usual theory of viscosity. The actual measurement of radiation pressure on a small sphere by means of a sensitive balance might supplement Biquard's observations which were made by an optical method. Computations of the type made in the present section indicate that radiation pressures may be easily measurable in this manner. For high precision the exact formula (74) should be employed, corrected for the compressibility of the sphere and the effect of viscosity on the scattered wave. A consideration of these points must, however, be left to a future investigation.

* 'C. R. Acad. Sci. Paris,' vol. 197, pp. 309–311 (1933).

Section 13—Summary and Conclusions

1. The pressure variation in a compressible medium in which the pressure is any function of the density is given by

$$\delta p = \rho_0 \dot{\phi} + \tfrac{1}{2}\frac{\rho_0}{c^2}\dot{\phi}^2 - \tfrac{1}{2}\rho_0 q^2. \tag{i}$$

The velocity potential ϕ is the solution of the wave equation with appropriate boundary conditions. The differentiation with respect to the time implied in $\dot{\phi}$ refers to an origin at rest. If $D\phi/Dt$ refers to an origin having components of velocity $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$,

$$\dot{\phi} = D\phi/Dt + u\dot{\xi} + v\dot{\eta} + w\dot{\zeta}, \tag{ii}$$

where, as usual $(u, v, w) = -\operatorname{grad}\phi$, are the velocity components referred to these axes.

2. To second-order terms, the time average of the acoustic radiation pressure on a sphere is given by

$$\overline{P} = \overline{P}_\phi + \overline{P}_q + \overline{P}_\zeta, \tag{iii}$$

where $\overline{P}_\phi$ and $\overline{P}_q$ are contributed by the second and third terms of (i), and $\overline{P}_\zeta$ is contributed by the last three terms of (ii) arising from the motion of the origin at the centre of the spherical obstacle which performs small oscillations under the influence of the first-order radiation pressure.

3. In radially symmetrical radiation fields for which the velocity potential can be expanded as a series of spherical wave functions in the form,

$$\phi_i = \sum_{n=0}^{\infty} A_n \psi_n(\kappa r) \,.\, (\kappa r)^n P_n(\mu), \tag{iv}$$

a general expression is obtained for the radiation pressure including the three terms of equation (iii).

4. In a plane progressive wave for which $\phi_i = Ae^{-i\kappa x}$ the radiation pressure $\overline{P}$ is always positive. For spheres whose circumference is small compared to the wave-length

$$\frac{\overline{P}}{\pi a^2} \sim 4(\kappa a)^4 F(\rho_0/\rho_1)\,\overline{E} \tag{v}$$

where $\overline{E}$ is the mean total energy-density in the medium and $F(\rho_0/\rho_1)$ is the relative density factor

$$F(\rho_0/\rho_1) = \frac{1 + \frac{2}{9}(1 - \rho_0/\rho_1)^2}{(2 + \rho_0/\rho_1)^2}. \tag{vi}$$

5. In plane stationary waves for which $\phi_i = A \cos \kappa (z + h)$, the radiation pressure $\overline{P}$ is periodic, depending on the position of the centre of the sphere with respect to the planes of loops and nodes. For small spheres

$$\frac{\overline{P}}{\pi a^2} = 2 (\kappa a) \sin 2\kappa h \, F (\rho_0/\rho_1) \, \overline{E} \tag{vii}$$

where $F (\rho_0/\rho_1)$ is the relative density factor

$$F (\rho_0/\rho_1) = \frac{1 + \frac{2}{3} (1 - \rho_0/\rho_1)}{2 + \rho_0/\rho_1}. \tag{viii}$$

It is important to note that in this case the radiation pressure in a stationary wave is of a much higher order of magnitude than in a progressive wave.

6. In general, the radiation pressure on a small sphere does not depend on the local specifications of the field, but on the nature of the field as a whole as related to the mode of generation of the sound waves.

7. The motion of a small sphere in a stationary plane radiation field is discussed in the light of equation (vii) above. When its mass is sufficiently great so that viscous resistance is negligible, the equations of motion are similar to that of a simple pendulum, oscillations taking place across the loops for relatively dense spheres ($\rho_0/\rho_1 < 2 \cdot 5$), and across loops for relatively light spheres ($\rho_0/\rho_1 > 2 \cdot 5$). The " time of formation " of striations is of the order $\frac{1}{4}$T, where T is the time of a complete oscillation, and is independent of the radius of the spherical particles.

For small particles in water, T is a minimum for the density ratio $\rho_1/\rho_0 = 1 \cdot 3$. In a supersonic radiation field of moderate intensity, $\frac{1}{4}T_{min}$ is of the order of 1 second.

For particles of cork-dust in stationary air-waves having a velocity amplitude of 31·4 cm/sec at a frequency of 500, $\frac{1}{4}$T is of the order of 7 seconds.

It appears probable that radiation pressure plays an important part in the formation of the familiar " dust striations " observed in resonance tubes.

8. When the particles are extremely small, so that the motion is principally controlled by viscosity, the trajectories of spherules falling vertically in a stationary radiation field having vertical nodal planes are obtained, and shown to be independent of the radius. If the particles are initially equally distributed over a horizontal plane, the density distribution after falling a depth z is determined. For relatively dense particles the density distribution has minima $n_0 e^{-\beta z}$ at the nodes and

maxima $n_0 e^{\beta z}$ at the loops, where β depends on the amplitude, wave-length, and the relative density, but is independent of the radius. In a supersonic stationary radiation field for which the velocity amplitude is 1·53 cm/sec at 570,000 cycles, the ratio of maximum to minimum density of the particle after falling through 5 cm is 257·2 : 0·0039, thus accounting for the very sharp striations observed by Boyle, Lehmann, and Reid.

9. The suspension of small particles and bubbles against gravity in a vertical stationary radiation field is discussed.

10. The results of the present investigation may be used in the design of torsion balances of optimum sensitivity for the measurement of amplitudes in stationary supersonic radiation fields.

The Energies of Alpha, Beta, and Gamma Rays

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This paper contains a continuation of the discussion of the energies of alpha, beta, and gamma rays on the theory previously suggested by the writer.* According to this theory the energies of beta and gamma rays are equal to $nq + \Sigma N_m E_m$ where $q = 3 \cdot 85$, $n = 0, 1, 2, 3 \ldots\ldots$, E_m denotes an electronic energy level and $N_m = 0, 1, 2, 3 \ldots\ldots\ldots$ The unit 10^5 electron volts will be used throughout.

We should expect elements with the same atomic number to give rays of equal energies. The chance of the emission of a ray with any given possible energy must depend on the nucleus as well as on the electronic system so that the relative intensities of the rays from different elements, having the same atomic number, may not be the same. The different elements may emit nuclear gamma rays with different values of the integer n and with different intervals between the emissions.

The three elements radium B, thorium B, and actinium B have the same atomic number 82 so we should expect to find beta and gamma rays of the same energies from these elements. The beta rays from actinium (B + C) and those from thorium B and its products have been observed, but there is some doubt as to which of the observed rays

* 'Proc. Roy. Soc.,' A, vol. 144, p. 280 (1934), vol. 145, p. 447 (1934).